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Thermodynamics of magnetohydrodynamic flows with axial symmetry

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We present strategies based upon optimization principles in the case of the axisymmetric equations of magnetohydrodynamics (MHD). We derive the equilibrium state by using a minimum energy principle under the constraints of the MHD axisymmetric equations. We also propose a numerical algorithm based on a maximum energy dissipation principle to compute in a consistent way the nonlinearly dynamically stable equilibrium states. Then, we develop the statistical mechanics of such flows and recover the same equilibrium states giving a justification of the minimum energy principle. We find that fluctuations obey a Gaussian shape and we make the link between the conservation of the Casimirs on the coarse-grained scale and the process of energy dissipation. We contrast these results with those of two-dimensional hydrodynamical turbulence where the equilibrium state maximizes a *H* function at fixed energy and circulation and where the fluctuations are nonuniversal.

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I. INTRODUCTION

The recent success of two experimental fluid dynamos [1,2] has renewed the interest in the mechanism of dynamo saturation, and thus of equilibrium configurations in magnetohydrodynamics (MHD). At the present time, there is no general theory to tackle this problem, besides dimensional theory. For example, in a conducting fluid with typical velocity V, density ρ, Reynolds number Re, and magnetic Prandtl number Pm, the typical level of magnetic field reached at saturation is necessarily [3]

$$B^2 = \mu_o \rho V^2 f(\mathrm{Re},\mathrm{Pm}), \tag{1}$$

where f is *a priori* an arbitrary function of Re and Pm. Many numerical simulations [4] lead to $f=1$, i.e., equipartition between the magnetic and turbulent energy. This is therefore often taken as a working tool in astrophysical or geophysical application. However, this result is far from applying to any saturated dynamo. Moreover, it does not give any information about possible anisotropy of the saturated field. It would therefore be interesting to build robust algorithms to derive the function f. By robust, we mean algorithms which depend on characteristic global quantities of the system (like total energy) but not necessarily on small-scale dissipation, or boundary conditions.

An interesting candidate in this regard is provided by statistical mechanics. In the case of pure fluid mechanics, statistical mechanics has mainly been developed within the frame of Euler equation for a two-dimensional perfect fluid. Onsager [5] used a Hamiltonian model of point vortices. Within this framework, two-dimensional (2D) turbulence is a state of negative temperature leading to the coalescence of vortices of the same sign [6]. Further improvement was provided by Robert and Sommeria [7] and Miller *et al.* [8] who independently introduced a discretization of the vorticity in a certain number of levels to account for the continuous nature of vorticity. Using the maximum entropy formalism of statistical mechanics [9], it is then possible to give the shape of the (meta)equilibrium solution of Euler's equation as well as the fine-grained fluctuations around it [10]. This is similar to Lynden-Bell's theory of violent relaxation [11] in stellar dynamics (see Chavanis [12] for a description of the analogy between 2D vortices and stellar systems). The predictive power of the statistical theory is, however, limited by the existence of an infinite number of constants, the Casimirs, which appears due to the particle-relabeling symmetry [13]: when going from the Lagrangian formulation (or Hamiltonian which is the most relevant approach from the statistical mechanics point of view) to the Eulerian one (which is the simplest formulation from the fluid mechanics point of view), the memory from the initial positions of the fluid particles is lost and the particle can be labeled in many ways. From Noether's theorem, this invariance is associated to the vorticity conservation and a Casimir is the integral of any function of the vorticity. The existence of this infinite set of constants precludes the finding of a universal distribution of fluctuations. In addition, the metaequilibrium state strongly depends on the *details* of the initial condition, not simply on the robust constraints such as circulation and energy. In certain occasions, for instance when the flow is forced at small scales, it may be more relevant to fix a prior distribution of vorticity fluctuations instead of the Casimirs [14]. Then, the coarse-grained flow maximizes a "generalized" entropy functional determined by the prior distribution of vorticity [15,16]. This approach may be particularly useful in the case of complex flows when there is a balance between forcing and dissipation at small scales. The situation is quite different in the case of MHD flows. The statistical mechanics of MHD flows has been recently explored by Jordan and Turkington [17] in two dimensions. In contrast with nonmagne-

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tized 2D hydrodynamics they obtained a *universal* Gaussian shape for the fluctuations. This comes from the fact that the Casimirs in the MHD case are integral quantities of the primitive velocity and magnetic fields and thus, in the continuum limit, have vanishing fluctuations. Therefore they do not alter the Gaussian distribution of fluctuations which is due to the quadratic nature of energy.

The pure 2D situation, however, seldom applies to astrophysical or geophysical flows. In this respect, it is interesting to develop statistical mechanics of systems closer to natural situations, albeit sufficiently simple so that the already well tested recipes of statistical mechanics apply. These requirements are met by flows with axial symmetry. Most natural objects are rotating, selecting this peculiar symmetry. Moreover, upon shifting from 2D to axisymmetric flows, one mainly shifts from a translation invariance along one axis, toward a rotation invariance along one axis. Apart from important physical consequences which need to be taken into account (for example, conservation of angular momentum instead of vorticity or momentum, curvature terms), this induces a similarity between the two systems which enables a natural adaptation of the 2D case to the axisymmetric case. This is shown in the present paper, where we recover the Gaussian shape of the fluctuations and make the link between the conservation of the Casimirs on the coarse-grained scale and the process of energy dissipation.

In the first part of the paper, we study the equilibrium shape by using a minimum energy principle under the constraints of the MHD axisymmetric equations. We also propose a numerical algorithm based on a maximum energy dissipation principle to compute in a consistent way the equilibrium states. This is similar to the relaxation equation proposed by Chavanis [15,16] in 2D hydrodynamics to construct stable stationary solutions of the Euler equation by maximizing the production of a H function while conserving the robust constraints (energy, circulation,…). Then, we develop the statistical mechanics of such flows and recover these equilibrium states, thereby providing a physical justification for the minimum energy principle.

II. MHD FLOWS WITH AXIAL SYMMETRY

A. Equations and notations

Consider the ideal incompressible MHD equations:

$$\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P + (\nabla\times\mathbf{B})\times\mathbf{B},$$

$$\partial_t \mathbf{B} + (\mathbf{U}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{U}, \tag{2}$$

where $\mathbf{U}$ is the fluid velocity, P is the pressure, $\sqrt{\rho\mu_0}\mathbf{B}$ is the magnetic field, and ρ is the (constant) fluid density. In the axisymmetric case we consider, it is convenient to introduce the poloidal/toroidal decomposition for the fields $\mathbf{U}$ and $\mathbf{B}$:

$$\mathbf{U} = \mathbf{U_p} + \mathbf{U_t} = \mathbf{U_p} + U\mathbf{e}_\theta, \tag{3}$$

$$\mathbf{B} = \mathbf{B_p} + \mathbf{B_t} = \nabla\times(A\mathbf{e}_\theta) + B\mathbf{e}_\theta,$$

where $\mathbf{A}=\mathbf{A_p}+A\mathbf{e}_\theta$ is the potential vector. This decomposition will be used in our statistical mechanics approach.

When considering energy methods, we shall introduce alternate fields, built upon the poloidal and toroidal decomposition. They are $\sigma_u=rU$, $\sigma_b=rA$, $\xi_u=\omega/r$, and $\xi_b=B/r$, where ω is the toroidal part of the vorticity field. In these variables, the ideal incompressible MHD equations (2) become, in the axisymmetric approximation, a set of four scalar equations:

$$\partial_t\sigma_b + \{\psi,\sigma_b\} = 0, \tag{4}$$

$$\partial_t\xi_b + \{\psi,\xi_b\} = \left\{\sigma_b,\frac{\sigma_u}{2y}\right\},$$

$$\partial_t\sigma_u + \{\psi,\sigma_u\} = \{\sigma_b,2y\xi_b\},$$

$$\partial_t\xi_u + \{\psi,\xi_u\} = \partial_z\left(\frac{\sigma_u^2}{4y^2}-\xi_b^2\right) - \{\sigma_b,\Delta_*\sigma_b\},$$

where the fields are function of the axial coordinate z and the modified radial coordinate $y=r^2/2$ and ψ is a stream function: $\mathbf{U_p}=\nabla\times(\psi/r\mathbf{e}_\theta)$. We have introduced a Poisson Bracket: $\{f,g\}=\partial_y f\partial_z g-\partial_z f\partial_y g$. We also defined a pseudo-Laplacian in the new coordinates:

$$\Delta_* = \frac{\partial^2}{\partial y^2} + \frac{1}{2y}\frac{\partial^2}{\partial z^2}. \tag{5}$$

Following Jordan and Turkington [17], we will make an intensive use of the operators (for more details, see the Appendix): curl which gives the toroidal part of the curl of any vector and **Curl** which takes a toroidal field as argument and returns the poloidal part of the curl. If $j=\mathrm{curl}\,\mathbf{B}$ is the toroidal part of the current and $\psi=r\,\mathrm{Curl}^{-1}(\mathbf{U_p})$, the following relations hold:

$$\xi_u = -\Delta_*\psi \quad \text{and} \quad j/r = -\Delta_*\sigma_b. \tag{6}$$

Under the shape (4), the ideal axisymmetric MHD equations of motion lead to the immediate identification of $\sigma_b=rA$ as a conserved quantity associated to axial symmetry. In the MHD case, it is thus the magnetic potential which plays the role of vorticity in the hydrodynamical case. The Casimirs will thus be functions of this conserved quantity as we now show.

B. Conservation laws

1. General case

The whole set of conservation laws of the axisymmetric ideal MHD equations have been derived by Woltjer [18]:

$$E = \frac{1}{2}\int\left\{\xi_u\psi - \sigma_b\Delta_*\sigma_b + \frac{\sigma_u^2}{2y} + 2y\xi_b^2\right\}dydz. \tag{7}$$

$$H_m = 2\int \xi_b N(\sigma_b)dydz,$$

$$H_c = \int\{F(\sigma_b)\xi_u + \sigma_u\xi_b F'(\sigma_b)\}dydz,$$

$$I = \int C(\sigma_b)dydz,$$

$$L = \int \sigma_u G(\sigma_b) dydz,$$

where C, N, F, and G are arbitrary functions. One can check that these integrals are indeed constants of motion by using Eq. (4) and the following boundary conditions: $\sigma_b=\sigma_u=\xi_u=\xi_b=0$ on the domain frontier. To prove the constancy of the third integral, one has to assume that $F(0)=0$. The reader, familiar with the three "classical invariants," namely, the energy, the magnetic helicity and the cross helicity, may be surprised to see here five sets (four of them infinite) of constants of motion. First let us argue about the appearance of the two families which are not classicaly taken into account. As stated above, the Casimirs I appear because of the magnetic potential conservation which is itself linked with the particle-relabeling symmetry [13]. On the other hand, the angular momentum conservation (last line with $G=1$) is conserved because of the axial symmetry. Second, one can easily see that if $\int H(\sigma_u,\xi_u,\xi_b)dydz$ is a conserved quantity, then for every F, $\int H(\sigma_u,\xi_u,\xi_b)F(\sigma_b)dydz$ is also a conserved quantity because of the conservation of σ_b. Therefore the introduction of the axial symmetry transforms the usual invariants into *families* of invariants. The interpretation of these integrals of motion is easier when considering a special case, introduced by Chandrasekhar [19].

2. Chandrasekhar model

The conservation laws take a simpler shape when one considers only linear and quadratic conservation laws, such that $N(\sigma_b)=F(\sigma_b)=G(\sigma_b)=\sigma_b$ and $N(\sigma_b)=G(\sigma_b)=1$. The case $F(\sigma_b)=1$ is forbidden by the requirement that F should vanish at the origin (see above). In that case, the set of conserved quantities can be split in two families. The first one is made up with conserved quantities of the ideal MHD system, irrespective of the geometry:

$$H_m = 2\int \xi_b\sigma_b dydz = \int \mathbf{A}\cdot\mathbf{B}d\mathbf{x} = 2\int ABd\mathbf{x}, \qquad (8)$$

$$H_c = \int \{\sigma_b\xi_u + \sigma_u\xi_b\}dydz = \int \mathbf{U}\cdot\mathbf{B}d\mathbf{x},$$

$$E = \frac{1}{2}\int \left\{\xi_u\psi - \sigma_b\Delta_*\sigma_b + \frac{\sigma_u^2}{2y} + 2y\xi_b^2\right\}dydz = \frac{1}{2}\int (\mathbf{U}^2+\mathbf{B}^2)d\mathbf{x},$$

where H_m is the magnetic helicity, H_c is the cross helicity, and E is the total energy. Note that due to the Lorentz force, the kinetic helicity is not conserved, unlike in the pure hydrodynamical case. The other family of conserved quantities is made of the particular integrals of motion which appear due to *axisymmetry*:

$$I = \int C(\sigma_b)dydz = \int C(rA)d\mathbf{x}, \qquad (9)$$

$$H_m' = 2\int \xi_b dydz = \int \frac{B}{r}d\mathbf{x},$$

$$L = \int \sigma_u G(\sigma_b)dydz = \int r^2UBd\mathbf{x},$$

$$L' = \int \sigma_u dydz = \int rUd\mathbf{x}.$$

Apart from the angular momentum L', it is difficult to give any physical interpretation for the other quantities. The class of invariant I is called the Casimirs of the system (if one defines a noncanonical bracket for the Hamiltonian system, they commute, in the bracket sense, will all other functionals). The conservation laws found by Woltjer are mere generalizations of these quantities.

C. Formal nonlinear dynamical stability

1. General case

Following Woltjer [18], we show that the extremization of energy at fixed I, H_m, H_c, and L determines the general form of stationary solutions of the MHD equations. We argue that the solutions that *minimize* the energy are nonlinearly dynamically stable for the inviscid equations.

To make the minimization, we first note that each integral is equivalent to an infinite set of constraints. Following Woltjer, we introduce a complete set of functions and label these functions and the corresponding integrals with an index n. Then, introducing Lagrange multipliers for each constraint, to first order, the variational problem takes the form

$$\delta E + \sum_{n=1}^{+\infty}\{\alpha^{(n)}\delta I^{(n)} + \mu_m^{(n)}\delta H_m^{(n)} + \mu_c^{(n)}\delta H_c^{(n)} + \gamma^{(n)}\delta L^{(n)}\} = 0. \qquad (10)$$

Taking the variations on σ_b, ξ_b, σ_u, and ξ_u, we find

$$\Delta_*\sigma_b = -F'(\sigma_b)\Delta_*\psi + F''(\sigma_b)\sigma_u\xi_b + G'(\sigma_b)\sigma_u + 2N'(\sigma_b)\xi_b + C'(\sigma_b), \qquad (11)$$

$$2y\xi_b = -2N(\sigma_b) - F'(\sigma_b)\sigma_u,$$

$$\frac{\sigma_u}{2y} = -F'(\sigma_b)\xi_b - G(\sigma_b),$$

$$\psi = -F(\sigma_b),$$

where we have set $F(\sigma_b)=\Sigma_{n=0}^{+\infty}\mu_c^{(n)}F_n(\sigma_b)$ and similar notations for the other functions. This is the general solution of the incompressible axisymmetric ideal MHD problem [18]. In the general case, it is possible to express the three field σ_u, ξ_u, and ξ_b in terms of σ_b. Then the first equation of the above system leads a partial differential equation for σ_b to be solved to find the equilibrium distribution. Note that the extremization of the "free energy" $J=E+\alpha I+\mu_m H_m+\mu_c H_c+\gamma L$ yields the same equations as Eq. (11). Differences will appear on second-order variations as we discuss below.

If we consider a purely magnetic case by taking $F=G=0$, we get

$$\Delta_* \sigma_b = C'(\sigma_b) - 2\frac{N(\sigma_b)N'(\sigma_b)}{y},$$

$$\xi_b = -\frac{N(\sigma_b)}{y}, \quad \mathbf{U} = \mathbf{0}. \tag{12}$$

This equation is obtained by minimizing the energy $E=\frac{1}{2}\int(\mathbf{U}^2+\mathbf{B}^2)d\mathbf{x}$ while conserving the generalized magnetic helicity $H_m=2\int(B/r)N(rA)d\mathbf{x}$ and the Casimirs $I=\int C(rA)d\mathbf{x}$. It can be therefore seen as a Grad-Shafranov equation (p. 30 in Ref. [20]). The Laplacian Δ_*' used in Biskamp's book is related to ours by the following relation: $\Delta_*'=r^2\Delta_*$. If we take, furthermore, $N(\sigma_b)=\mu_m\sigma_b$ and $C(\sigma_b)=K\sigma_b$ we get

$$\Delta_* \sigma_b = K - 2\mu_m^2\frac{\sigma_b}{y},$$

$$\xi_b = -\mu_m\frac{\sigma_b}{y}, \quad \mathbf{U} = \mathbf{0}. \tag{13}$$

With $K=0$, we obtain the so-called Beltrami equation. It minimizes the energy $E=\frac{1}{2}\int(\mathbf{U}^2+\mathbf{B}^2)d\mathbf{x}$ while conserving the magnetic helicity $H_m=\int\mathbf{B}\cdot\mathbf{A}d\mathbf{x}$. This variational principle was suggested by Taylor [21]. In vectorial form it leads to $\nabla\times\mathbf{B}=-2\mu_m\mathbf{B}$. If we account also for the conservation of $I_0=\int rAd\mathbf{x}$, we get Eq. (13).

2. Chandrasekhar model

In the Chandrasekhar model, the arbitrary functions are at most linear functions of σ_b: $N(\sigma_b)=\mu_m\sigma_b+\mu_m'$, $F(\sigma_b)=\mu_c\sigma_b$ and $G(\sigma_b)=\gamma\sigma_b+\gamma'$. Thus the stationary profile in the Chandrasekhar model is given by

$$\Delta_* \sigma_b = -\mu_c\Delta_*\psi + \gamma\sigma_u + 2\mu_m\xi_b + C'(\sigma_b), \tag{14}$$

$$2y\xi_b = -2\mu_m\sigma_b - 2\mu_m' - \mu_c\sigma_u,$$

$$\frac{\sigma_u}{2y} = -\mu_c\xi_b - \gamma\sigma_b - \gamma',$$

$$\psi = -\mu_c\sigma_b.$$

From the previous equations, we obtain

$$2y(1-\mu_c^2)\xi_b = 2(\gamma\mu_c y - \mu_m)\sigma_b + 2\mu_c\gamma' y - 2\mu_m', \tag{15}$$

$$(1-\mu_c^2)\sigma_u = 2(\mu_c\mu_m - \gamma y)\sigma_b + 2\mu_c\mu_m' - 2\gamma' y,$$

$$\psi = -\mu_c\sigma_b,$$

where σ_b is given by the differential equation:

$$(1-\mu_c^2)^2\Delta_*\sigma_b = \Phi(\sigma_b) - \left(\frac{2\mu_m^2}{y} + 2\gamma^2 y\right)\sigma_b - 2\gamma\gamma' y - \frac{2\mu_m\mu_m'}{y}. \tag{16}$$

These expressions can be used to prove that these fields are stationary solutions of the axisymmetric MHD equations. We now turn to the stability problem. Since the functional $J=E+\alpha I+\mu_m H_m+\mu_c H_c+\gamma L$ is conserved by the ideal dynamics, a minimum of J will be nonlinearly dynamically stable in the *formal* sense of Holm *et al.* [22]. Note that this implication is not trivial because the system under study is dimensionally infinite. We admit that their analysis can be generalized to the present context. Since the integrals which appear in the functional J are conserved individually, a minimum of energy at fixed other constraints also determines a nonlinearly dynamically stable stationary solution of the MHD equations. This second stability criterion is more refined than the first (it includes it). We shall not prove these results, nor write the second order variations, here. We refer to Ellis *et al.* [14] for a precise discussion in the related context of 2D hydrodynamical flows.

If we ignore the conservation of angular momentum ($\gamma=\gamma'=0$) and the conservation of H_m' ($\mu_m'=0$), we get

$$(1-\mu_c^2)^2\Delta_*\sigma_b = \Phi(\sigma_b) - \frac{2\mu_m^2}{y}\sigma_b,$$

$$\xi_b = -\frac{\mu_m}{1-\mu_c^2}\frac{\sigma_b}{y}, \quad \mathbf{U} = -\mu_c\mathbf{B}. \tag{17}$$

In that case, the velocity and the magnetic field are aligned (see also Sec. III C 1). This solution is obtained by minimizing the energy $E=\frac{1}{2}\int(\mathbf{U}^2+\mathbf{B}^2)d\mathbf{x}$ while conserving the magnetic helicity $H_m=\int\mathbf{A}\cdot\mathbf{B}d\mathbf{x}$, the cross helicity $H_c=\int\mathbf{U}\cdot\mathbf{B}d\mathbf{x}$ and the Casimirs $I=\int C(rA)d\mathbf{x}$. The alignment of $\mathbf{U}$ and $\mathbf{B}$ can be obtained by minimizing the energy at fixed cross helicity. This variational principle was suggested by Matthaeus and Montgomery [23].

D. Numerical algorithm to construct stable equilibria

1. General case

It is usually difficult to solve directly the system of Eqs. (15) and (16) and make sure that they ensure a stable stationary solution of the MHD equations. Instead, we shall propose a set of relaxation equations which continuously decreases the energy while conserving any other integral of motion. This allows construction of solutions of the system (15) and (16) which are energy minima and respect the other constraints. A physical justification of this procedure linked to the dissipation of energy will be given in Sec. III C 1.

Our relaxation equations can be written under the generic form

$$\frac{\partial \sigma}{\partial t} = -\nabla \cdot \mathbf{J}_\sigma, \tag{18}$$

where σ stands for $\sigma_b, \xi_b, \sigma_u$, or ξ_u. Using straightforward integration by parts, we then get

$$\dot{I} = \int \mathbf{J}_{\sigma_b} \cdot \nabla [C'(\sigma_b)] dydz, \tag{19}$$

$$\dot{H}_m = 2 \int \{\mathbf{J}_{\xi_b} \cdot \nabla [N(\sigma_b)] + \mathbf{J}_{\sigma_b} \cdot \nabla [N'(\sigma_b)\xi_b]\} dydz,$$

$$\dot{H}_c = \int \{\mathbf{J}_{\xi_u} \cdot \nabla [F(\sigma_b)] + \mathbf{J}_{\sigma_b} \cdot \nabla [F'(\sigma_b)\xi_u + F''(\sigma_b)\sigma_u \xi_b] + \mathbf{J}_{\sigma_u} \cdot \nabla [F'(\sigma_b)\xi_b] + \mathbf{J}_{\xi_b} \cdot \nabla [F'(\sigma_b)\sigma_u]\} dydz,$$

$$\dot{L} = \int \{\mathbf{J}_{\sigma_u} \cdot \nabla [G(\sigma_b)] + \mathbf{J}_{\sigma_b} \cdot \nabla [G'(\sigma_b)\sigma_u]\} dydz,$$

$$\dot{E} = \int \left\{ \mathbf{J}_{\xi_u} \cdot \nabla \psi - \mathbf{J}_{\sigma_b} \cdot \nabla (\Delta_* \sigma_b) + \mathbf{J}_{\sigma_u} \cdot \nabla \left(\frac{\sigma_u}{2y}\right) + \mathbf{J}_{\xi_b} \cdot \nabla (2y\xi_b) \right\} dydz.$$

To construct the optimal currents, we rely on a procedure of maximization of the rate of dissipation of energy $\dot{E}$. This is similar to the maximization of the production of a H function used in Refs. [15,16] in 2D hydrodynamics. We thus maximize $\dot{E}$ given the conservation of $\dot{I} = \dot{H}_m = \dot{H}_c = \dot{L} = 0$. Such maximization can only have solution for bounded currents (if not, the fastest evolution is for infinite currents). Therefore we also impose a bound on J_σ^2 where, as before, σ stands for σ_b, ξ_b, σ_u, ξ_u.

Writing the variational problem under the form

$$\delta \dot{E} + \sum_{n=1}^{+\infty} \{\alpha^{(n)}(t)\delta \dot{I}^{(n)} + \mu_m^{(n)}(t)\delta \dot{H}_m^{(n)} + \mu_c^{(n)}(t)\delta \dot{H}_c^{(n)} + \gamma^{(n)}(t)\delta \dot{L}^{(n)}\} + \sum_\sigma \frac{1}{D_\sigma}\delta\left(\frac{J_\sigma^2}{2}\right) = 0 \tag{20}$$

and taking variations on $\mathbf{J}_{\sigma_b}, \mathbf{J}_{\xi_b}, \mathbf{J}_{\sigma_u}, \mathbf{J}_{\xi_u}$, we obtain the optimal currents. Inserting their expressions in the relaxation equations (18), we get

$$\frac{\partial \sigma_b}{\partial t} = \nabla \cdot \{D_{\sigma_b} \nabla [-\Delta_* \sigma_b + C'(\sigma_b, t) + 2\xi_b N'(\sigma_b, t) + \xi_u F'(\sigma_b, t) + \sigma_u \xi_b F''(\sigma_b, t) + G'(\sigma_b, t)\sigma_u]\}, \tag{21}$$

$$\frac{\partial \xi_b}{\partial t} = \nabla \cdot \{D_{\xi_b} \nabla [2y\xi_b + 2N(\sigma_b, t) + F'(\sigma_b, t)\sigma_u]\},$$

$$\frac{\partial \sigma_u}{\partial t} = \nabla \cdot \left\{ D_{\sigma_u} \nabla \left[\frac{\sigma_u}{2y} + \xi_b F'(\sigma_b, t) + G(\sigma_b, t)\right]\right\},$$

$$\frac{\partial \xi_u}{\partial t} = \nabla \cdot \{D_{\xi_u} \nabla [\psi + F(\sigma_b, t)]\},$$

where we have set $F(\sigma_b, t) = \sum_{n=0}^{+\infty} \mu_c^{(n)}(t) F_n(\sigma_b)$ and similar notations for the other functions. The time evolution of the Lagrange multipliers $\mu_c^{(n)}(t)$, etc., are obtained by substituting the optimal currents in the constraints $\dot{H}_c^{(n)} = 0$, etc., and solving the resulting set of algebraic equations. Using the expression of the optimal currents and the condition that $\dot{I} = \dot{H}_m = \dot{H}_c = \dot{L} = 0$, we can show that

$$\dot{E} = -\int \left\{ \frac{J_{\xi_u}^2}{D_{\xi_u}} + \frac{J_{\sigma_b}^2}{D_{\sigma_b}} + \frac{J_{\sigma_u}^2}{D_{\sigma_u}} + \frac{J_{\xi_b}^2}{D_{\xi_b}} \right\} dydz \leqslant 0, \tag{22}$$

provided that the diffusion currents D_{ξ_u}, D_{σ_b}, D_{σ_u}, and D_{ξ_b} are positive. Thus the energy decreases until all the currents vanish. In that case, we obtain the static equations (11). In addition, this numerical algorithm guarantees that only energy *minima* are reached; maxima or saddle points of energy are linearly unstable [15]. Note that if we fix the Lagrange multipliers instead of the constraints, the foregoing relaxation equations lead to a stationary state which minimizes the "free energy" J. Then, as stated above, the constructed solutions will be nonlinearly dynamical stable solutions of the MHD set of equations. However, forbidding the Lagrange multipliers to depend on time, we may "miss" some stable solutions of the MHD equations. Indeed, we know that minima of the free energyare nonlinearly stable solutions of the problem but we do not know if they are the only ones: some solutions can be minima of E at fixed I, H_m, H_c, and L while they are not minima of $J = E + \alpha I + \mu_m H_m + \mu_c H_c + \gamma L$. This is similar to a situation of ensemble inequivalence in thermodynamics; see Refs. [14,24].

2. Chandrasekhar model

In the Chandrasekhar model (with $\mu_m' = \gamma' = 0$), the previous equations can be simplified. Furthermore, as the equilibrium solution does not depend on the particular value of the diffusion coefficients (these are only multiplicative factors of the optimal currents), we set for simplicity $D_{\xi_u} = D_{\sigma_b} = D_{\sigma_u} = D_{\xi_b} = 1$. The relaxation equations then reduce to

$$\frac{\partial \sigma_b}{\partial t} = \Delta\{-\Delta_* \sigma_b + C'(\sigma_b, t) + 2\mu_m(t)\xi_b + \mu_c(t)\xi_u + \gamma(t)\sigma_u\}, \tag{23}$$

$$\frac{\partial \xi_b}{\partial t} = \Delta\{2y\xi_b + 2\mu_m(t)\sigma_b + \mu_c(t)\sigma_u\},$$

$$\frac{\partial \sigma_u}{\partial t} = \Delta\left\{\frac{\sigma_u}{2y} + \mu_c(t)\xi_b + \gamma(t)\sigma_b\right\},$$

$$\frac{\partial \xi_u}{\partial t} = \Delta\{\psi + \mu_c(t)\sigma_b\},$$

where the Lagrange multipliers evolve in time so as to conserve the constraints (19).

These equations are the MHD counterpart of the relaxation equations proposed by Chavanis [15,16] for 2D hydrodynamical flows described by the Euler equation. In this context, a nonlinearly dynamically stable stationary solution of the Euler equation maximizes a H function at fixed energy and circulation. A justification of this procedure, linked to the increase of H functions on the coarse-grained scale, will be further discussed in Sec. IV and compared with the MHD case.

III. STATISTICAL MECHANICS OF AXISYMMETRIC FLOWS

In the previous section, we obtained general equilibrium velocity and magnetic field *profiles* through minimization of the energy under constraints. In the present section, we derive velocity and magnetic field *distribution* using a thermodynamical approach, based upon a statistical mechanics of axisymmetric MHD flows. As we later check, the distribution we find are such that their mean fields obey the equilibrium profiles found by energy minimization. For simplicity, we focus here on the Chandrasekhar model.

A. Definitions and formalism

Following Miller [8], Robert [7], and Jordan and Turkington [17], we introduce a coarse-graining procedure through the consideration of a length scale under which the details of the fields are irrelevant. The microstates are defined in terms of all the microscopic possible fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$. On this phase space, we define the local distribution of velocity and magnetic field $\rho(\mathbf{x},\mathbf{u},\mathbf{b})$. This forms a macrostate. The coarse-grained field (denoted by a bar) is determined by the following relations:

$$\overline{\mathbf{U}}(\mathbf{x}) = \int \mathbf{u}\rho(\mathbf{x},\mathbf{u},\mathbf{b})d\mathbf{u}d\mathbf{b}, \tag{24}$$

$$\overline{\mathbf{B}}(\mathbf{x}) = \int \mathbf{b}\rho(\mathbf{x},\mathbf{u},\mathbf{b})d\mathbf{u}d\mathbf{b}.$$

We introduce the mixing entropy

$$S[\rho] = -\int \rho(\mathbf{x},\mathbf{u},\mathbf{b})\ln[\rho(\mathbf{x},\mathbf{u},\mathbf{b})]d\mathbf{x}d\mathbf{u}d\mathbf{b}, \tag{25}$$

which has the form of Shanon's entropy in information theory [9,25]. It is proportional to the logarithm of disorder, where the disorder is the number of microstates consistent with a given macrostate $\rho(\mathbf{x},\mathbf{u},\mathbf{b})$. The most probable macrostate $\rho^*(\mathbf{x},\mathbf{u},\mathbf{b})$ maximizes the entropy subject to the constraints. The mathematical ground for such a procedure is that an overwhelming majority of all the possible microstates with the correct values for the constants of motion will be close to this state (see Ref. [7] for a precise definition of the neighborhood of a macrostate and the proof of this concentration property). Note that this approach gives not only the coarse-grained field $(\overline{\mathbf{U}},\overline{\mathbf{B}})$ but also the fluctuations around it through the distribution $\rho(\mathbf{x},\mathbf{u},\mathbf{b})$.

Each conserved quantity has a numerical value which can be calculated given the initial condition, or from the detailed knowledge of the fine-grained fields. The integrals calculated with the coarse-grained quantities are not necessarily conserved because part of the integral of motion can go into fine-grained fluctuations (as we shall see, this is the case for the energy in MHD flows). This induces a distinction between two classes of conserved quantities, according to their behavior through coarse graining. Those which are not affected are called robust, whereas the other one are called fragile.

B. Constraints

In this section, it is convenient to come back to the original velocity and magnetic fields. The constraints are the coarse-grained values of the conserved quantities (8). The key point, as noted by Jordan and Turkington [17], is that the quantity coming from a spatial integration of one of the fields $\mathbf{u}$ or $\mathbf{b}$ is smooth. In our case, it amounts to neglecting the fluctuations of A which is spatially integrated from $\mathbf{B}$ and write $A=\overline{A}$. Thus the coarse-grained values of the conserved quantities are given by

$$\overline{I} = \int C(r\overline{A})d\mathbf{x}, \tag{26}$$

$$\overline{H_m} = 2\int \overline{AB}d\mathbf{x},$$

$$\overline{H_c} = \int \mathbf{u}\cdot\mathbf{b}\rho(\mathbf{x},\mathbf{u},\mathbf{b})d\mathbf{x}d\mathbf{u}d\mathbf{b},$$

$$\overline{E} = \frac{1}{2}\int (\mathbf{u}^2+\mathbf{b}^2)\rho(\mathbf{r},\mathbf{u},\mathbf{b})d\mathbf{x}d\mathbf{u}d\mathbf{b},$$

$$\overline{H'_m} = 2\int \frac{\overline{B}}{r}d\mathbf{x},$$

$$\overline{L} = \int \overline{AU}r^2 d\mathbf{x},$$

$$\overline{L'} = \int \overline{U}r d\mathbf{x}.$$

The constraint $\overline{I}$ is the Casimir, connected to the conservation of σ_b along the motions. In the present case, it is a robust quantity as it is conserved on the coarse-grained scale. As stated previously, the quantities $\overline{H_m}$, $\overline{H_c}$, and $\overline{E}$ are the mean values of the usual quadratic invariants of ideal MHD, namely the magnetic helicity, the cross helicity, and the energy. On the contrary, the quantities $\overline{H'_m}$, $\overline{L}$, and $\overline{L'}$ are specific to axisymmetric systems. Because these last three conservation laws are usually disregarded in classical MHD theory, it is interesting in the sequel to separate the study in two cases, according to which the conservation of $\overline{H'_m}$, $\overline{L}$, and

$\overline{L'}$ is physically relevant ("rotating case") or is not physically relevant ("classical case").

C. Gibbs state

1. Classical case

The MHD equations develop a mixing process leading to a metaequilibrium state on the coarse-grained scale. It is obtained by maximizing the mixing entropy $S[\rho]$ with respect to the distribution ρ at fixed $\overline{I}$, $\overline{H_m}$, $\overline{H_c}$, and $\overline{E}$ (we omit the bars in the following). We have

$$\delta S = -\int (1 + \ln \rho)\delta\rho d\mathbf{x}d\mathbf{u}d\mathbf{b}, \tag{27}$$

$$\delta H_c = \int \mathbf{u}\cdot\mathbf{b}\,\delta\rho d\mathbf{x}d\mathbf{u}d\mathbf{b},$$

$$\delta E = \frac{1}{2}\int (\mathbf{u}^2 + \mathbf{b}^2)\delta\rho d\mathbf{x}d\mathbf{u}d\mathbf{b}.$$

The variation of the magnetic helicity and the Casimirs is more tedious because they involve the coarse-grained field $\overline{A}$. For the magnetic helicity, we have

$$\delta H_m = 2\int (\delta\overline{AB} + \overline{A}\delta\overline{B})d\mathbf{x}. \tag{28}$$

Now, using an integration by parts, it is straightforward to show that

$$\int \delta AB d\mathbf{x} = \int \delta\mathbf{B}_P\cdot\mathbf{A}_P d\mathbf{x}. \tag{29}$$

Therefore

$$\delta H_m = 2\int (\delta\overline{\mathbf{B}}_P\cdot\overline{\mathbf{A}}_P + \overline{A}\delta\overline{B})d\mathbf{x} = 2\int \overline{\mathbf{A}}\cdot\delta\overline{\mathbf{B}}d\mathbf{x}$$
$$= 2\int \overline{\mathbf{A}}\cdot\mathbf{b}\,\delta\rho d\mathbf{x}d\mathbf{u}d\mathbf{b}. \tag{30}$$

Regarding the variation of the Casimirs, we find

$$\delta I = \int C'(r\overline{A})r\delta\overline{A}d\mathbf{x} = \int C'(r\overline{A})r\,\mathrm{Curl}^{-1}\,\overline{\mathbf{B}}_P d\mathbf{x}$$
$$= \int \mathbf{curl}^{-1}[rC'(r\overline{A})]\cdot\delta\overline{\mathbf{B}}_P d\mathbf{x}, \tag{31}$$

or

$$\delta I = \int \mathbf{curl}^{-1}[rC'(r\overline{A})]\cdot\mathbf{b}_P\delta\rho d\mathbf{x}d\mathbf{u}d\mathbf{b}. \tag{32}$$

Writing the variational principle in the form

$$\delta S - \beta\delta E - \mu_m\delta H_m - \mu_c\delta H_c - \sum_{n=1}^{+\infty}\alpha^{(n)}\delta I^{(n)} = 0, \tag{33}$$

we find that

$$1 + \ln\rho = -\frac{\beta}{2}(\mathbf{u}^2 + \mathbf{b}^2) - 2\mu_m\overline{\mathbf{A}}\cdot\mathbf{b} - \mu_c\mathbf{u}\cdot\mathbf{b}$$
$$- \mathbf{curl}^{-1}[rC'(rA)]\cdot\mathbf{b}_P. \tag{34}$$

It is appropriate to write $\mathbf{u}=\overline{\mathbf{U}}+\mathbf{u}'$ and $\mathbf{b}=\overline{\mathbf{B}}+\mathbf{b}'$ where the first term denotes the coarse-grained field. Then, Eq. (34) can be rewritten

$$1 + \ln\rho = -\frac{\beta}{2}(u'^2 + b'^2) - \mu_c\mathbf{u}'\cdot\mathbf{b}' - \mu_m\overline{\mathbf{A}}\cdot\overline{\mathbf{B}} - \frac{\mu_c}{2}\overline{\mathbf{U}}\cdot\overline{\mathbf{B}}$$
$$- \left(\frac{\overline{\mathbf{U}}}{2} + \mathbf{u}'\right)\cdot[\beta\overline{\mathbf{U}} + \mu_c\overline{\mathbf{B}}] - \left(\frac{\overline{\mathbf{B}}}{2} + \mathbf{b}'\right)$$
$$\cdot[\beta\overline{\mathbf{B}} + 2\mu_m\overline{\mathbf{A}} + \mu_c\overline{\mathbf{U}} + \mathbf{curl}^{-1}[rC'(rA)]\ . \tag{35}$$

Hence the fluctuations are Gaussian:

$$\rho = \frac{1}{Z}\exp\left\{-\frac{\beta}{2}(\mathbf{u}'^2 + \mathbf{b}'^2) - \mu_c\mathbf{u}'\cdot\mathbf{b}'\right\}$$
$$= \frac{1}{Z}\exp\left\{\frac{1}{2}\sum_{i,j}x_iA_{ij}x_j\right\}, \tag{36}$$

where we defined a six-dimensional vector: $x_i=(u_1',u_2',u_3',b_1',b_2',b_3')$. The mean field is given by

$$\beta\mathbf{U} + \mu_c\mathbf{B} = \mathbf{0}, \tag{37}$$

$$\beta B + 2\mu_m A + \mu_c U = 0,$$

$$\beta\mathbf{B}_P + 2\mu_m\mathbf{A}_P + \mu_c\mathbf{U}_P + \mathbf{curl}^{-1}[rC'(rA)] = 0.$$

Taking the curl of these relations and using curl $\mathbf{B}_P=j$, curl $\mathbf{U}_P=\omega$, and curl $\mathbf{A}_P=B$, we recover the equilibrium distribution (14) with $\gamma=\gamma'=\mu_m'=0$. Therefore, in this classical case, the equilibrium profiles are such that mean velocity and mean magnetic field are aligned. This is a well-known feature of turbulent MHD, which has been observed in the solar wind (where $\mathbf{v}\approx\pm\mathbf{B}$). It has been linked with a principle of minimum energy at constant cross helicity (see Chap. 7.3 of Ref. [20] and references therein). This feature is also present in numerical simulation of decaying 2D MHD turbulence, where the current and the vorticity are seen to be very much equal [26]. This can therefore be seen as the mere outcome of conservation of quadratic integral of motions, and may provide an interesting general rule about dynamo saturation in systems where these quadratic constraints are dominant.

Using the Gaussian shape for the fluctuations, it is quite easy to derive the mean properties of the fluctuations. To do so, we will make use of the following standard results [27]:

$$Z = (2\pi)^3\sqrt{\det[A]} = (2\pi)^3[\beta^2 - \mu_c^2]^{3/2}, \quad \langle x_ix_j\rangle = (A^{-1})_{ij}. \tag{38}$$

Then, it is easy to show that part of the energy is going into the fluctuations and that there is equipartition between the fluctuating parts of the magnetic energy and of the kinetic energy:

$$\langle u'^2\rangle = \langle b'^2\rangle = \frac{3\beta}{\beta^2-\mu_c^2}. \tag{39}$$

One can also calculate the quantity of cross helicity going into the fluctuations:

$$\langle \mathbf{u}'\cdot\mathbf{b}'\rangle = -\frac{3\mu_c}{\beta^2-\mu_c^2}. \tag{40}$$

One should notice that there is no net magnetic helicity in the fluctuations because A is strictly conserved: $\langle \mathbf{a}'\cdot\mathbf{b}'\rangle=0$. Then, the fractions of magnetic energy, cross helicity, and kinetic energy going into the fluctuations are

$$\frac{\langle b'^2\rangle}{\int \overline{B}^2 d\mathbf{x}} = \frac{\langle \mathbf{u}'\cdot\mathbf{b}'\rangle}{\int \overline{U}\cdot\overline{B}d\mathbf{x}} = \frac{3\beta}{\beta^2-\mu_c^2}\mathcal{M}^{-1}, \tag{41}$$

$$\frac{\langle u'^2\rangle}{\int \overline{U}^2 d\mathbf{x}} = \frac{\beta^2}{\mu_c^2}\frac{3\beta}{\beta^2-\mu_c^2}\mathcal{M}^{-1},$$

where $\mathcal{M}=\int\overline{B}^2 d\mathbf{x}$ is the magnetic energy of the coarse-grained field. The first equation shows that there is an equal fraction of magnetic energy and cross helicity into the fluctuations while the positivity of the magnetic energy requires $\beta^2>\mu_c^2$. Using this inequality and the second line, we see that the fraction of kinetic energy going into the fluctuations is then bigger than that of the magnetic energy and cross helicity. This gives some mathematical ground to the energy minimization procedure we used in Sec. II C.

2. Rotating case

The situation changes when the other constants of motion are taken into account. We have

$$\delta H_m' = 2\int \frac{b}{r}\delta\rho d\mathbf{x}d\mathbf{u}d\mathbf{b}, \tag{42}$$

$$\delta L' = \int ur\,\delta\rho d\mathbf{x}d\mathbf{u}d\mathbf{b}.$$

On the other hand,

$$\begin{aligned}\delta L &= \int (\overline{\delta A U} + \overline{A}\,\overline{\delta U})r^2 d\mathbf{x} = \int (\overline{U}\,\mathbf{curl}^{-1}\ \overline{\delta \mathbf{B}}_P + \overline{A}\,\overline{\delta U})r^2 d\mathbf{x}\\ &= \int (\mathbf{curl}^{-1}(r^2\overline{U})\cdot\overline{\delta\mathbf{B}}_P + \overline{A}\,\overline{\delta U}r^2)d\mathbf{x}\\ &= \int (\mathbf{curl}^{-1}(r^2\overline{U})\cdot\mathbf{b}_P + \overline{A}ur^2)\delta\rho d\mathbf{x}d\mathbf{u}d\mathbf{b}.\end{aligned} \tag{43}$$

Adding Lagrange multipliers $-\mu_m'$, $-\gamma$, and $-\gamma'$ for H_m', L, and L', respectively, we find that the expression (34) is multiplied by

$$\exp\left\{-2\mu_m'\frac{b}{r} - \gamma' ru - \gamma[\mathbf{curl}^{-1}(r^2\overline{U})\cdot\mathbf{b}_P + \overline{A}ur^2]\right\}. \tag{44}$$

The distribution of fluctuations is then still Gaussian and given by Eq. (36) but now the mean-field equations are

$$\beta\mathbf{U}_P + \mu_c\mathbf{B}_P = \mathbf{0}, \tag{45}$$

$$\beta U + \mu_c B + \gamma' r + \gamma A r^2 = 0,$$

$$\beta B + 2\mu_m A + \mu_c U + \frac{2\mu_m'}{r} = 0,$$

$$\beta\mathbf{B}_P + 2\mu_m\mathbf{A}_P + \mu_c\mathbf{U}_P + \mathbf{curl}^{-1}[rC'(rA)] + \gamma\,\mathbf{curl}^{-1}(r^2U) = \mathbf{0}.$$

Taking the curl of the vectorial relations, we get the system (14).

3. Application to the magnetic field of stars

Therefore the relation between the velocity and the magnetic field is not linear anymore, when taking into account additional constants of motion. The linearity is only valid for the poloidal component: $\mathbf{B}_P\propto\mathbf{U}_P$. The toroidal component obeys

$$\beta\left(U + \frac{\gamma'}{\beta}r\right) = -\mu_c B - \gamma A r^2. \tag{46}$$

We can interprete $U+\gamma'/\beta r$ as the relative velocity around a solid rotation $\Omega=-\gamma'/\beta$. Indeed, γ' is the Lagrange multiplier for the angular momentum constraint. The nontrivial term responsible for the departure from linearity is $-\gamma A r^2$. Thus the breaking of the proportionality between the velocity and the magnetic field can be attributed to the conservation of the angular momentum in the Chandrasekhar model. This is an interesting feature because this conservation rule is likely to be more relevant in rapidly rotating objects. This may explain the dynamo saturation in rotating stars, where linearity between magnetic and velocity field is observed for slowly rotating stars and is broken for rotator faster than a certain limit. To illustrate such a behavior, we used astrophysical data [28] giving the bolometric luminosity L_b, the period P, the mass M, and the color index $B-V$ for a given set of stars (of the dwarf type). From this data, an estimate of the toroidal parts of the magnetic and velocity fields are as follows.

For the magnetic field, it is done in two steps:

1. From the value of the color index (or equivalently the temperature), we use the Hertzsprung-Russell Diagram to fit the value of the star radius R in order to obtain the correct value for the luminosity.

2. Once we know the radius of the star, we use an empirical power law [29] linking the luminosity with the magnetic flux Φ:

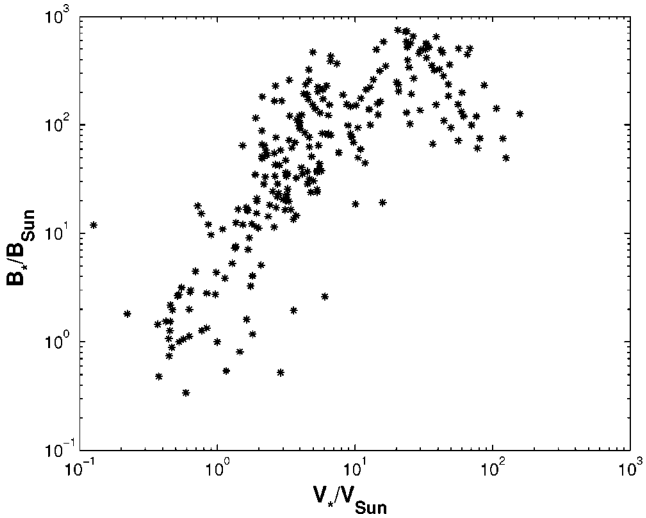

FIG. 1. Magnetic field of stars (of late-type dwarfs) calculated from their x-ray emission and compared to that of the Sun, vs their rotation velocity divided by that of the Sun.

$$L_b \propto \Phi^{1.15} \sim (BR^2)^{1.15} \tag{47}$$

and thus we calculate the magnetic field from the luminosity. For the velocity field, we assume that the interior of the star is close to a solid body rotation and derive the velocity from the rotation period P: $U \propto 2\pi R/P$. In Fig. 1, we have plotted the magnetic field B versus the velocity field (both quantity being expressed in units of the corresponding solar value). One sees that the magnetic field is a linear function of the velocity field for the slow rotation rate but saturates for the higher rotation rate, as predicted by Eq. (46). Note that the nonproportionality between velocity and magnetic field can also be due to additional conserved quantities such as those considered by Woltjer [18].

D. Relaxation equations

The previous discussion has shown that the energy of the coarse-grained field has the tendency to decrease while the other invariants of the inviscid axisymmetric MHD equations are approximately conserved. Thus the energy is a fragile invariant, while the others are robust. Using this observation, we build up a system of relaxation equations which provide a small-scale parametrization of axisymmetric MHD turbulence. Coarse graining the axisymmetric MHD equations, we obtain a system of equations of the form

$$\partial_t \overline{\sigma_b} + \{\Psi, \overline{\sigma_b}\} = -\nabla \cdot \mathbf{J}_{\sigma_b}, \tag{48}$$

$$\partial_t \overline{\xi_b} + \{\Psi, \overline{\xi_b}\} = \left\{ \overline{\sigma_b}, \frac{\overline{\sigma_u}}{2y} \right\} - \nabla \cdot \mathbf{J}_{\xi_b},$$

$$\partial_t \overline{\sigma_u} + \{\Psi, \overline{\sigma_u}\} = \{\overline{\sigma_b}, 2y\overline{\xi_b}\} - \nabla \cdot \mathbf{J}_{\sigma_u},$$

$$\partial_t \overline{\xi_u} + \{\Psi, \overline{\xi_u}\} = \partial_z \left(\frac{\overline{\sigma_u}^2}{4y^2} - \overline{\xi_b}^2 \right) - \{\overline{\sigma_b}, \Delta_* \overline{\sigma_b}\} - \nabla \cdot \mathbf{J}_{\sigma_b},$$

where the currents take into account the correlations of the fine-grained fluctuations. Using a heuristic principle, we propose to determine the currents so as to maximize the rate of energy dissipation while conserving the other constraints. This is similar to the idea of the maximum entropy production principle of Robert and Sommeria [30] in 2D turbulence stating that the evolution toward equilibrium is such that it maximizes the entropy creation under the given constraints. This is equivalent to say that the evolution toward the equilibrium state is the fastest. The calculation are exactly the same as in Sec. II D 1 and lead to the system of equations (21) in the general case and Eq. (23) in the case of the Chandrasekhar model with additional advective terms for the coarse-grained quantities. However, the interpretation of the new system of equations is different from the previous one: we obtain here a system of equations for the coarse-grained variables, the fine-grained ones being parametrized, in the relaxation currents, in terms of coarse-grained quantities.

If we set the coarsed-grained part of the velocity field to zero ($\overline{\sigma_u} = \overline{\xi_u} = \psi = 0$), we get a system of equations

$$\frac{\partial \overline{\sigma_b}}{\partial t} = \nabla \cdot \{D_{\sigma_b} \nabla [-\Delta_* \overline{\sigma_b} + C'(\overline{\sigma_b}, t) + 2\overline{\xi_b} N'(\overline{\sigma_b}, t)]\}, \tag{49}$$

$$\frac{\partial \overline{\xi_b}}{\partial t} = \nabla \cdot \{D_{\xi_b} \nabla [2y\overline{\xi_b} + 2N(\overline{\sigma_b}, t)]\},$$

linking the poloidal part ($\overline{\sigma_b}$) and the toroidal part ($\overline{\xi_b}$) of the magnetic field. This system of equations relaxes toward the stationary state described by the Grad-Shafranov equation (12). Note that we have set the *coarse-grained* part of the velocity field to zero. Thus Eqs. (49) describe the organization of magnetic field by purely *fluctuating* velocity field in the spirit of a "turbulent dynamo." In this respect, it may be of interest to stress the analogies and the differences between the relaxation equations (49) and the mean-field equations of MHD dynamo. When considering the effect of a *fluctuating* velocity field (assumed to be isotropic) on the process of magnetic field generation, see Steenbeck *et al.* [31], the equation for the mean (or coarse-grained) magnetic field is found to be

$$\partial_t \overline{\mathbf{B}} = \nabla \times [\alpha(\overline{\mathbf{B}})\overline{\mathbf{B}}] - \nabla \times [\beta(\overline{\mathbf{B}}) \nabla \times (\overline{\mathbf{B}})]. \tag{50}$$

Assuming that the mean magnetic field is axisymmetric, the foregoing equation can be rewritten with the scalar variables as

$$\frac{\partial \overline{\sigma_b}}{\partial t} = 2\beta y \Delta_* \overline{\sigma_b} + 2\alpha y \overline{\xi_b}, \tag{51}$$

$$\frac{\partial \overline{\xi_b}}{\partial t} = \boldsymbol{\nabla} \cdot \left[\frac{\beta}{2y} \boldsymbol{\nabla} (2y\overline{\xi_b}) - \frac{\alpha}{2y} \boldsymbol{\nabla} \overline{\sigma_b} \right].$$

The first term in the right-hand side is a turbulent (magnetic) diffusivity whereas the second one is the so-called α effect which builds up coarse-grained magnetic field from a fluctuating velocity field. In the case of an axisymmetric mean magnetic field to which Eqs. (51) apply, the coupling between the toroidal ($\overline{\xi_b}$) and poloidal ($\overline{\sigma_b}$) part of the magnetic field is proportional to the coefficient α. In the "kinematic approximation," where the effect of the Lorentz force is neglected, the coefficient α is constant and proportional to the kinetic helicity $H_k \propto \overline{\mathbf{u}' \cdot \boldsymbol{\nabla} \times \mathbf{u}'}$ of the fluctuating velocity field. In this sense, it is related to purely hydrodynamical variables. However, taking into account the retroaction of the magnetic field on the velocity field, Pouquet *et al.* [32] were able to write a nonlinear α effect as a difference between the kinetic and the magnetic helicity spectra: $\alpha \sim (H_k - H_m)$. It is interesting to note that the second equation in Eq. (49) has a structure similar to the second equation in Eq. (51), especially in the case where $N(\sigma_b) = \mu_m \sigma_b$. In that case, the equivalent of the α parameter is the Lagrange multiplier μ_m associated to the conservation of magnetic helicity. This is an important feature that these two systems have in common, namely the coupling between the poloidal and the toroidal part of the magnetic field is proportional to a quantity linked with the magnetic helicity: in Eq. (51), the α effect can be expressed in terms of the spectrum of the magnetic helicity (in the nonlinear regime) whereas in Eq. (49), μ_m is the Lagrange multiplier associated to the conservation of magnetic helicity. Our relaxation equations therefore recover the fact that the equilibrium configuration of the coarse-grained magnetic field is mainly monitored by the magnetic helicity. This is due to the fact that magnetic helicity experiences an inverse cascade (from small to large scales) in MHD turbulence as has been shown by Ref. [33] and as it is evidenced in our study by the fact that there is no net magnetic helicity in the fluctuations (see the end of Sec. III C 1).

It is also of interest to compare the first equation in Eqs. (49) and (51). Despite some analogies, these two equations differ in the sense that the right-hand side of the first equation in Eq. (49) is written as the divergence of a current. This current respects the conservation of all the Casimirs, and in particular the conservation of $\int \overline{\sigma_b} d\mathbf{x}$. This is a consequence of the assumed *complete* axisymmetry of the system. By constrast, the first equation in Eq. (51) does not conserve the Casimirs, nor the integral of $\overline{\sigma_b}$. This is due to the fact that only the mean field is assumed to be axisymmetric, not the microfields (or fluctuations). Therefore the first equation in Eqs. (49) and (51) will behave differently. In particular, an initial axisymmetric magnetic field cannot decay or grow following the dynamics of Eqs. (49). Thus this parametrization does not describe a dynamo mechanism in the usual sense, contrary to Eq. (51). It just describes a *reorganization* of the magnetic field by turbulence. In fact, Eqs. (49) parametrize a mixing process on the coarse-grained scale, just as in 2D turbulence (chaotic mixing) or stellar dynamics (phase mixing) [12]. This observation can be related to Cowling's theorem [34] which precludes an axisymmetric magnetic field to grow by dynamo action. In Cowling's theorem, there is viscosity and the magnetic field finally goes to zero. In our case, the flow is inviscid and the magnetic field is reorganized by turbulence, without being dissipated. This gives rise to a coherent state, like the one described by the Grad-Shafranov equation or by the more general mean-field equations derived in Sec. III C 1. In the presence of a small viscosity, these coherent structures would be observed during a long, transient, stage of the dynamics.

IV. SUMMARY

We have developed a statistical theory of axisymmetric MHD equations generalizing the 2D approach by Jordan and Turkington [17]. We derived the velocity and magnetic field distribution and established the differential equations determining the equilibrium profiles for the mean flow. Like in the 2D case, the fluctuations around the mean field are found Gaussian, a universal feature connected to the conservation of the Casimirs under the coarse graining together with the quadratic nature of energy. The equilibrium profiles are characterized by an alignment of the velocity and magnetic field, which is broken when the angular momentum conservation is taken into account. The statistical equilibrium profiles are found to correspond to profiles obtained under minimization of energy subject to the constraints. Thus, in the MHD case, in the presence of a coarse graining (or a small viscosity), the energy is dissipated while the helicity, the angular momentum, *and* the Casimirs are approximately conserved (hydromagnetic selective decay). In particular, the energy of the coarse-grained field decreases: $\overline{E} = \frac{1}{2}\int \overline{U^2 + B^2} d\mathbf{x} \neq \frac{1}{2}\int (\overline{U}^2 + \overline{B}^2) d\mathbf{x} = E_{c.g.}$ because part of total energy goes into fine-grained fluctuations $E_{fluct} = \overline{E} - E_{c.g.}$. Therefore the metaequilibrium state minimizes $E_{c.g.}$ (fragile) at fixed I, H_m, H_c, and L (robust). This can be justified in the "classical case" (Sec. III C 1) where we showed that the fraction of kinetic energy going into the fluctuating part of the fields was higher than that of the other quantities, namely the magnetic energy and the cross helicity. The "rotating case" (Sec. III C 2) requires more algebra and is left for further study.

In contrast, in the 2D hydrodynamical case, the Casimirs are fragile quantities (because they are expressed as function of the vorticity which is not an integral quantity as the magnetic potential is) and thus are altered by the coarse-graining procedure. This is true in particular for a special class of Casimirs $S = -\int C(\omega) d\mathbf{x}$, called H functions, constructed with a convex function C such that $C'' > 0$ [15,35]. This leads to two very different behaviors of hydrodynamical turbulence compared to the hydromagnetic one. First, the H functions $S_{c.g.} = -\int C(\overline{\omega}) d\mathbf{x}$ calculated with the coarse-grained vorticity $\overline{\omega}$ increase with time, a property similar to the H theorem of thermodynamics, while the circulation and energy are approximately conserved (hydrodynamic selective decay). However, this property is true for an infinite number of functionals and their increase is not necessarily monotonic (it can only be proved that $S_{c.g.}(t) \geqslant S_{c.g.}(0)$, see Ref. [35]). Because of this generalized selective decay, the metaequilibrium state maximizes one of the H functions $S_{c.g.}$ (fragile) at fixed E and Γ (robust). For example, Chavanis and Sommeria [10] showed that in the limit of strong mixing (or for Gaussian fluctuations), the quantity to maximize is minus the enstrophy, giving some mathematical basis to an (inviscid) "minimum enstrophy principle." In this context, the enstrophy of the coarse-grained flow decreases: $\overline{\Gamma}_2 = \int \overline{\omega^2} d\mathbf{x} \neq \int \overline{\omega}^2 d\mathbf{x} = \Gamma_2^{c.g.}$ because part of total enstrophy goes into fine-grained fluctuations $\Gamma_{fluct} = \overline{\Gamma}_2 - \Gamma_2^{c.g.}$. In fact, this property is true for any H function: $\overline{S} = \int \overline{C(\omega)} d\mathbf{x} \neq \int C(\overline{\omega}) d\mathbf{x} = S_{c.g.}$. Therefore the H function that is maximized at metaequilibrium is *nonuniversal* (it is not necessarily the enstrophy) and can take a wide diversity of forms as discussed by Chavanis [15,16]. Due to their resemblance with entropy functionals (they increase with time, one is maximum at metaequilibrium with fixed robust constraints,…), and because they generally differ from the Boltzmann functional $S_B = -\int \omega \ln \omega d\mathbf{x}$, the H functions are sometimes called "generalized entropies" [15]. However, this is only an *analogy* with thermodynamics because they cannot be obtained from a combinatorial analysis and none of them is singled out by the Euler equation (nonuniversality) [36]. From the statistical mechanics point of view, the relevant mixing entropy $S[\rho]$ is the Boltzmann entropy (for an ensemble of levels), but there is an infinite number of Casimir invariants (depending on the microscale fields) to take into account when deriving the Gibbs state. Consequently, the shape of the fluctuations is not universal. This is why the H function that is maximized at metaequilibrium is also nonuniversal. However, if the distribution of fluctuations is imposed by some external mechanism (e.g., a small-scale forcing) as suggested by Ellis *et al.* [14], the functional $S[\overline{\omega}]$ is now a well-determined functional selected by the Gibbs state and the prior vorticity distribution [14,15]. In that case, it really has the status of a generalized entropy in the sense of large deviations as its maximization (at fixed circulation and energy) determines the optimal coarse-grained vorticity field via the Cramer formula.

Our computation can provide interesting insight regarding dynamo saturation. It is, however, limited by its neglect of dissipation and forcing mechanism. It would therefore be interesting to generalize this kind of approach to more realistic systems. In that case, the entropy might not be the relevant quantity anymore, but rather the turbulent transport, or the entropy production [37].

APPENDIX: CURL OPERATORS

Following Jordan and Turkington [17], we define

$$\text{curl}\,\mathbf{B} = (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{e}_\theta, \tag{A1}$$

$$\mathbf{Curl}\,A = \boldsymbol{\nabla} \times (A\mathbf{e}_\theta),$$

for any vector $\mathbf{B}$ and scalar A. It is straightforward to show that we have the following relations:

$$\text{curl}\,\mathbf{Curl}\left(\frac{A}{r}\right) = -r\Delta_* A, \tag{A2}$$

$$\int A\,\text{curl}\,\mathbf{B} d\mathbf{x} = \int \mathbf{Curl}\,A \cdot \mathbf{B} d\mathbf{x}. \tag{A3}$$

Setting $A = \text{Curl}^{-1}\,\mathbf{B}'$ and $\text{curl}\,\mathbf{B} = A'$ in the last identity, we get

$$\int \text{Curl}^{-1}\,\mathbf{B}' A' d\mathbf{x} = \int \mathbf{B}' \cdot \mathbf{curl}^{-1}\,A' d\mathbf{x}. \tag{A4}$$

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